$$w = W + A_1 w_1^* + A_2 w_2^*$$
$$f = -\tfrac{1}{2}(x^2+y^2)pR + A_1 f_1^* + A_2 f_2^* \tag{18}$$

where

$$w_1^* = h\cos\left(q_o \frac{x}{R}\right)$$
$$f_1^* = -ERh^2 q_o^{-2}\cos\left(q_o \frac{x}{R}\right)$$
$$w_2^* = h\sin\left(\tfrac{1}{2}q_o \frac{x}{R}\right)\sin\left(\frac{(3)^{1/2}}{2} q_o \frac{y}{R}\right) \tag{19}$$
$$f_2^* = -ERh^2 q_o^{-2}\sin\left(\tfrac{1}{2}q_o \frac{x}{R}\right)\sin\left(\frac{(3)^{1/2}}{2} q_o \frac{y}{R}\right)$$

A substitution of Eq. (18) into Eq. (2), using the appropriate Hamiltonian Functional of shallow shells, gives

$$H = \int_{t_1}^{t_2}\left\{\frac{h^4\rho_s}{4}\left[\dot{A}_1^{\,2} + \tfrac{1}{2}\dot{A}_2^{\,2} + \frac{g_s+\rho a}{\rho_s h}\underline{(\dot{A}_1 A_1 + \tfrac{1}{2}\dot{A}_2 A_2)}\right] + \frac{Eh^3}{R}\left[\tfrac{1}{2}\left(1-\frac{p}{p_b}\right)A_1^{\,2} - \frac{9c}{32}A_1 A_2^{\,2} + \tfrac{1}{4}\left(1-\frac{p}{p_b}\right)A_2^{\,2} - \frac{p}{p_b}A_1\mu_1 - \frac{p}{p_b}A_2\mu_2\right\} S\,dt$$

where S is the area of the shell. The Euler equations are

$$\frac{1}{\omega^2}\left[\ddot{A}_1 + \frac{g_s+\rho a}{\rho_s h}A_1\right] + \left(1-\frac{p}{p_b}\right)A_1 = \frac{p}{p_b}u_1 + \frac{9c}{32}A_2^{\,2}$$
$$\frac{1}{\omega^2}\left[\ddot{A}_2 + \frac{g_s+\rho a}{\rho_s h}A_2\right] + \left(1-\frac{p}{p_b}\right)A_2 = \frac{p}{p_b}\mu_2 + \frac{9c}{8}A_2^{\,2} \tag{20}$$

where

$$\omega^2 = (2/R)E/\rho_s$$

III. Numerical Results and Comparison with Experiments

The series of shallow spherical caps tested in Ref. 1 has been examined analytically. The shells were made of clear Vinylite (PVC) with $\rho_s = 0.062$ lb/in.^{3}, $E = 0.475 \times 10^6$ psi, and $\nu = 0.3$, and were clamped at the end of a shock tube. The pressure pulse was generated by rarefaction wave.

In the analysis, the static bifurcation load in its corresponding bifurcation mode and all the quantities defined in Eq. (9) are first determined by the finite-element technique developed in Ref. 4. The imperfection amplitude is estimated from the experimental buckling load of static test. Equation (9) is then solved by direct numerical integration for various pulse magnitudes of 2.2 msec duration. Such a pulse duration is equal to 0.7 to 2.4 periods of the shells considered at $p = 0$. Dynamic instability is defined by the phenomenon that large maximum amplitudes A are produced by small increases in pressure p.

Figure 1 gives the comparison of the one-mode theoretical and the experimental results. It can be seen that the present theory generally gives a lower bound to the experimental data. If the acoustic damping is neglected in the analysis, the theoretical dynamic buckling load will be lower. Computation has also been carried out using local buckling theory for several λ's. The buckling load is generally 10 ~ 20% lower than those given in Fig. 1.

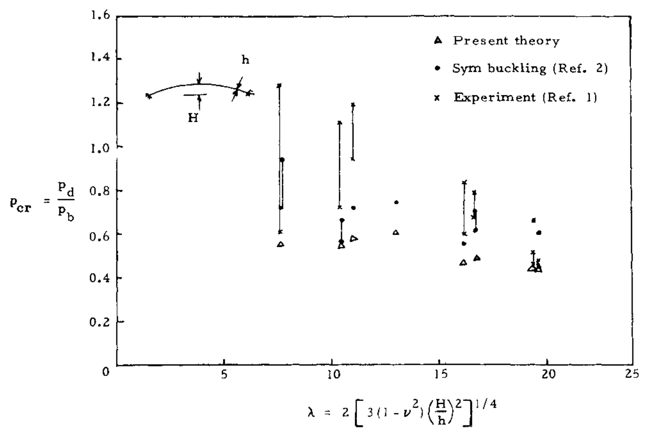

Fig. 1 Dynamic buckling pulse pressure of duration 2.2 m.

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A Programing Approach to Optimal Structural Design Using Structural Indices

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Introduction

EVER since Michell's classic paper on optimum truss geometry[1] the potential and value of structural efficiency generalizations have been recognized by engineer and theorist alike. Prior to the introduction of the digital computer, investigators employed a parametric approach with algebraic manipulation being the means of appraising relative structural efficiency. Apparently first coined by Wagner,[2] the term "structural index" has played an important role in these parametric approaches. The use of structural indices has historically been associated with the simultaneous mode design technique advanced by Wagner,[2] Farrar,[3] and Shanley[4] and later employed extensively by Gerard[5] and Cox[6] and many other investigators, as can be seen in the recent bibliographies.[7,8] The limitations of simultaneous mode design, which result from the necessary assumptions on the activity of constraints, have led to a recognition that constraints must be expressed as inequalities in the general problem formulation[9] and the attitude that a structural index solution is accordingly forfeit. "Slack variables" have been used recently by the present writer[10] and a similar

Presented as Paper 73-344 at the AIAA/ASME/SAE 14th Structures, Structural Dynamics, and Materials Conference, Williamsburg, Va., March 20–22, 1973; submitted April 10, 1973; revision received December 7, 1973.

Index categories: Aircraft Structural Design (Including Loads); Computer Technology and Computer Simulation Techniques.

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method earlier by Krzyś and Życzkowski[11] in an effort to overcome these limitations. These studies have shown that structural index solutions are possible for the simpler problems without assertions concerning the activity of constraints. There is great difficulty, however, in applying these methods where complex and numerous failure constraints exist[12] and certainly any parametric method will break down where algebraic transcendence prevents merit and constraint combination. The availability of computer algorithms for the mathematical programing solution to optimal design problems has led to the solution of the more complex problems by automated methods[9,13-15] and the attitude that structural index results are generally unattainable.

It is the intention of the present study to demonstrate that mathematical programing techniques can be employed to evaluate structural index solutions where unwieldy constraints frustrate an algebraic solution. Specifically, where constraints are too complex or numerous to be algebraically combined with merit, the programing formulation itself can be reduced to a nondimensional index form. This programing problem, when automated and solved by iterative search on the computer, yields a generalized solution (for a specified material selection) which can be graphically represented in terms of the structural indices of the environment.

Approach to a Programing Formulation Using Structural Indices

Weight Merit

Consider the general weight function

$$W = \rho A L \tag{1}$$

where ρ is the material density, A the cross-sectional area, and L the member length. Dividing both sides of Eq. (1) by L^3 yields

$$W/L^3 = \rho(A/L^2) \tag{2}$$

Define W/L^3 as the weight index merit function and note that its minimization is equivalent to least weight as L is parametrically fixed. It follows from Eq. (2) that the weight index merit function must construct in a nondimensional form of the design variable since A/L^2 is dimensionless and ρ is independent of the design variables. Hence, employing functional notation, it can be said that

$$W/L^3 = W/L^3(\rho, Xi/L) \tag{3}$$

where the Xi are the cross-sectional design variables.

Stress Constraints

Consider the stress constraint

$$\sigma_A \leqq \sigma_F \tag{4}$$

where σ_A is some general applied stress and σ_F a general failure stress strength or stability. The expression for σ_A depends on the fixed load environment and the design variables. Let P be the generalized load in lbs and L the member length. Therefore

$$\sigma_A = \sigma_A(P, L, Xi) \tag{5}$$

Suppose the above expression for σ_A is reconstructed in terms of dimensionless variables Xi/L by substitution for each Xi the equivalent $(Xi/L)L$. Dimensionally Eq. (5) must take the form

$$\sigma_A = \sigma_A(P/L^2, Xi/L) \tag{6}$$

which states that the expression for applied stress will depend on the load environment as some lumped index with the units of stress when constructed in terms of the design variables in nondimensional form.†

Now consider the dimensional form of σ_F which generally depends on modulus of elasticity and yield stress, E, σ_y, member length, L, and the design variables, Xi

$$\sigma_F = \sigma_F(E, \sigma_y, L, Xi) \tag{7}$$

As before, if Eq. (7) is reconstructed in a nondimensional form of the Xi, it follows that

$$\sigma_F = \sigma_F(E, \sigma_y, Xi/L) \tag{8}$$

since the units of E or σ_y are lb/in^2. Combining the generalizations of Eqs. (6) and (8) with Eq. (4) yields

$$\sigma_A\left(\frac{P}{L^2}, \frac{Xi}{L}\right) \leqq \sigma_F\left(E, \sigma_y, \frac{Xi}{L}\right) \tag{9}$$

which expresses the generality for formulation of stress constraints in nondimensional index form.

Displacement Constraints

Consider the displacement constraint

$$\delta \leqq \delta_{max} \tag{10}$$

where δ is some displacement of interest with the dimensions of inches and δ_{max} is the fixed bound displacement. The displacement function will generally depend on the fixed load environment, the material stiffness, E, and the design variables, Xi, as shown

$$\delta = \delta(P, L, E, Xi) \tag{11}$$

Replacing each Xi with $(Xi/L)\,L$ does not yield a dimensionless index form since Eq. (11) becomes

$$\delta = \delta\left(\frac{P/L^2}{E}, L, \frac{Xi}{L}\right) \tag{12}$$

due to the dimensions of δ being those of L. However, the quantity δ/L is dimensionless and Eq. (12) in this form becomes

$$\frac{\delta}{L} = \frac{\delta}{L}\left(\frac{P/L^2}{E}, \frac{Xi}{L}\right) \tag{13}$$

Dividing both sides of Eq. (10) by L and combining with the generalization of Eq. (13) yields

$$\frac{\delta}{L}\left(\frac{P/L^2}{E}, \frac{Xi}{L}\right) \leqq \frac{\delta_{max}}{L} \tag{14}$$

which expresses the generality for formulation of deflection constraints in nondimensional index form based on a bound displacement defined as a fraction of the member length.

Mathematical Programing Formulation

Given the fixed generalized load and length for a structural element P, L† and material properties ρ, E, σ_y, let Xi be the design variables. Define $\{Xi\}_{opt}$ such that

$$W(Xi) \to \min \tag{15}$$

subject to

$$\sigma_A \leqq \sigma_F$$

and

$$\delta \leqq \delta_{max}$$

Based on the generalizations of Eqs. (3, 9, and 14), the preceding formulation can be reconstructed as: given the structural index P/L^2 and material properties ρ, E, σ_y, let Xi/L be the design variables. Define $\{Xi/L\}_{opt}$ such that

$$W/L^3(\rho, Xi/L) \to \min$$

subject to

$$\sigma_A\left(\frac{P}{L^2}, \frac{Xi}{L}\right) \leqq \sigma_F\left(E, \sigma_y, \frac{Xi}{L}\right) \tag{16}$$

and

$$\frac{\delta}{L}\left(\frac{P/L^2}{E}, \frac{Xi}{L}\right) \leqq \frac{\delta_{max}}{L}$$

The solution to Eqs. (16) can be effected by computer evaluation for parametric selections of structural index with the result that

† Other load types are represented here dimensionally in terms of P and L, e.g., a line load, q, in units of lb/in. has dimensions P/L and therefore P/L^2 in Eq. (6) represents q/L for this case. In combined loads this dimensional structure must be maintained, in which case P/L^2 must represent the several distinct structural indices involved, i.e., P/L^2, q/L, M/L^3, etc.

$\{Xi/L\}_{\text{opt}}$ and $\{W/L^3\}_{\text{opt}}$ are evaluated as a function of structural index for a specified material selection and bound values of $\delta_{\max}/L$. As previously explained,† the structural index P/L^2 in Eqs. (16) is the stress units generalization of any structural load type or combination.

The following example will illustrate the set up and solution in the form of Eqs. (16) for a combined loads case.

Beam Column Example

Algebraic index solutions to combined loads optimization problems are particularly difficult to attain because of the unwieldy stress and displacement constraints. Employing simultaneous mode design and a combination of algebraic and graphical procedures, Burns and Skogh[16] present a parametric solution to the combined compression and shear of stiffened plates based on a procedure suggested by Switzky.[17] Felton and Dobbs,[18] employing similar parametric procedures, present a solution to combined bending and torsion of circular tubes.

To illustrate the general programing method being reported, consider the weight merit and constraints[19] for the circular tube beam column of Fig. 1.

Weight merit

$$W = \rho\pi D t L \rightarrow \min \tag{17}$$

Local buckling

$$\frac{P}{\pi D t} + \frac{Q\left(\dfrac{8P}{\pi E D^3 t}\right)^{1/2}}{4P} ED \tan\left\{\left(\frac{2P}{\pi E D^3 t}\right)^{1/2} L\right\} \leqq 0.40E\left(\frac{t}{D}\right) \tag{18}$$

Yielding

$$\frac{P}{\pi D t} + \frac{Q\left(\dfrac{8P}{\pi E D^3 t}\right)^{1/2}}{4P} ED \tan\left\{\left(\frac{2P}{\pi E D^3 t}\right)^{1/2} L\right\} \leqq \sigma_y \tag{19}$$

Deflection $\left(\dfrac{\delta_{\max}}{L} \leqq 0.01\right)$

$$\frac{Q}{2P\left(\dfrac{8P}{\pi E D^3 t}\right)^{1/2} L}\left\{\tan\left[\left(\frac{2P}{\pi E D^3 t}\right)^{1/2} L\right] - \left(\frac{2P}{\pi E D^3 t}\right)^{1/2} L\right\} \leqq 0.01 \tag{20}$$

By replacing D and t with $(D/L)L$ and $(t/L)L$ in Eqs. (17–20) nondimensional variables will be formed with the automatic generation of the two structural indices P/L^2 and Q/L^2 as shown below.

Weight merit

$$W/L^3 = \rho\pi(D/L)(t/L) \rightarrow \min \tag{21}$$

Local buckling

$$\frac{\dfrac{P}{L^2}}{\pi\dfrac{D}{L}\dfrac{t}{L}} + \frac{\dfrac{Q}{L^2}\left[\dfrac{8P/L^2}{\pi E(D/L)^3(t/L)}\right]^{1/2} E\left(\dfrac{D}{L}\right)}{4P/L^2} \times \tan\left\{\left[\frac{2P/L^2}{\pi E\left(\dfrac{D}{L}\right)^3\left(\dfrac{t}{L}\right)}\right]\right\}^{1/2} \leqq 0.40E\frac{\left(\dfrac{t}{L}\right)}{\left(\dfrac{D}{L}\right)} \tag{22}$$

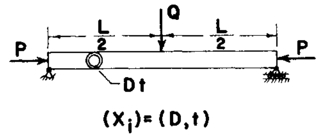

Fig. 1 Variable description for beam-column.

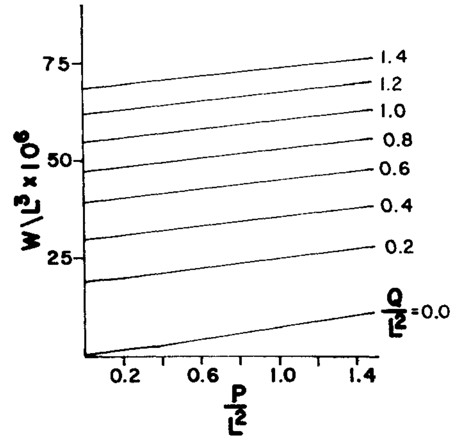

Fig. 2 Beam-column weight index vs P/L^2 and Q/L^2 for mild steel.

Yielding

$$\frac{\dfrac{P}{L^2}}{\pi\dfrac{D}{L}\dfrac{t}{L}} + \frac{\dfrac{Q}{L^2}\left[\dfrac{8P/L^2}{\pi E(D/L)^3(t/L)}\right]^{1/2} E\left(\dfrac{D}{L}\right)}{4P/L^2} \times \tan\left\{\left[\frac{2P/L^2}{\pi E\left(\dfrac{D}{L}\right)^3\left(\dfrac{t}{L}\right)}\right]\right\}^{1/2} \leqq \sigma_y \tag{23}$$

Deflection

$(\delta_{\max}/L \leqq 0.01)$

$$\frac{Q/L^2}{2\dfrac{P}{L^2}\left[\dfrac{8P/L^2}{\pi E\left(\dfrac{D}{L}\right)^3\dfrac{t}{L}}\right]^{1/2}}\left\{\tan\left[\frac{2P/L^2}{\pi E\left(\dfrac{D}{L}\right)^3\dfrac{t}{L}}\right]^{1/2} - \left[\frac{2P/L^2}{\pi E\left(\dfrac{D}{L}\right)^3\dfrac{t}{L}}\right]^{1/2}\right\} \leqq 0.01 \tag{24}$$

A numerical evaluation of Eqs. (21–24) using mild steel for parametric selections of the load indices, P/L^2 and Q/L^2 yields the optimum weight index and dimensionless variable plots shown in Figs. 2 and 3. Figure 4 (shown for the case of $Q/L^2 = 0.4$) illustrates the activity of the local buckling and yield constraints with a considerable margin against failure by excessive deflection ($\delta/L \leqq 0.01$). Similar results were observed at all other considered values of Q/L^2 ($Q/L^2 \leqq 2.0$) leading to the generalization that

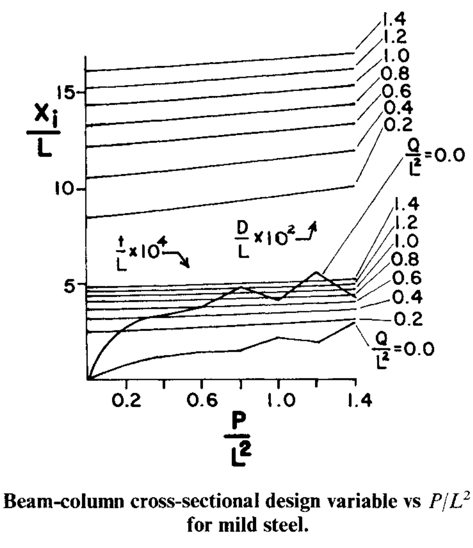

Fig. 3 Beam-column cross-sectional design variable vs P/L^2 and Q/L^2 for mild steel.

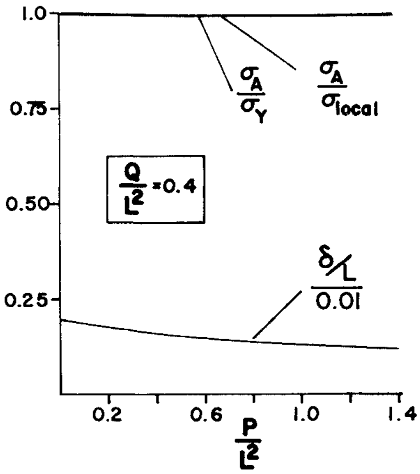

Fig. 4 Activity of constraints for mild steel beam column.

under suppression of local buckling and yielding the optimum mild steel tube has active local and yield constraints with inherent stiffness considerably in excess of a 1% deflection limitation. The discontinuous behavior of t/L and D/L where $P/L^2 < 0.35$ for the case $Q/L^2 = 0$ in Fig. 3 accompanied by continuity in weight merit (Fig. 2) shows the existence of multiple optima for the simple column beyond the load index for which the yield constraint becomes active. The corresponding inactivity of either the local or Euler modes or a combination gives rise to a family of cross sections each having the property of least weight. Although no unique pair (t, D) can be found from the programed solution, their product is computable so as to achieve the yielding area, and then either the local or Euler constraints can be set equal to yield stress to effect a unique sizing.

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Floating Piecewise Linear Approximation of a Nonlinear Constitutive Equation

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Introduction

ONE of the main difficulties arising in large deflection analysis of structures has its origin in the nonlinearity of the stress-strain relations. This difficulty is usually overcome by piecewise linearization of the constitutive equation. A method for construction of the "best" piecewise linear approximation is proposed. It will be shown, through an examination of a simple model, that the suggested method yields a sequence of upper bounds to the load-deflection behavior. The maximum load carrying capacities for the given model obtained by the approximate and the exact methods are the same.

Exact Solution to a Simple Model

Consider the simple model shown in Fig. 1. This model is taken from Ref. 1 where an approximate analysis has been given. The model consists of a rigid bar AB and a spring BC. The stress strain relation of the spring is given by

$$\theta - \theta_o = (\sigma/E) + K(\sigma/E)^n \tag{1}$$

where σ is the stress in the spring and θ_o is an initial imperfection. Equation (1) is of the type given in Ref. 2. The static equilibrium requirement, for small value of θ ($tg\theta \approx \theta$) is

$$A\sigma = P\theta \tag{2}$$

where A is the area of the spring and P the external applied load. By substitution of Eq. (2) into Eq. (1), the exact load-deflection behavior is obtained

$$\theta - \theta_o = (P\theta/EA) + K(P\theta/EA)^n \tag{3}$$

The behavior of the perfect system is obtained from Eq. (3) by taking $\theta_o = 0$. The resulting nonlinear equation has two branches

$$\theta = 0 \quad \text{and} \quad (P/EA)[1 + K(P\theta/EA)^{n-1}] = 1 \tag{4}$$

Received May 29, 1973. This work is based on a part of a thesis to be submitted to the Technion—Israel Institute of Technology, Haifa, in partial fulfilment of the requirements for the Degree of Doctor of Science. The research reported in this Note has been sponsored in part by the Air Force Office of Scientific Research, U.S. Air Force, under Grant 72-2394. The authors would like to thank E. Shaish of the Department of Aeronautical Engineering, Technion, for his assistance in the computational work.

Index category: Structural Stability Analysis.

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